Parcijalni izvodi f-ja više promjenjivih

Posmatrajmo f-ju z dvije promjenjive $z=f(x,y)$.
Parcijalni izvod po x-u označavamo sa z'_x ili sa $\frac{\partial z}{\partial x}$ (delta z po delta x) ili sa f'_x i definišemo

$$z'_x=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

Parcijalni izvod po y-nu označavamo sa z'_y ili sa $\frac{\partial z}{\partial y}$ (∂-delta) ili sa f'_y i definišemo

$$z'_y=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

(#) Odrediti parcijalne izvode f-ja

a) $z = x^3 + 5xy^2 - y^3$

b) $u = \frac{x}{y} + \frac{y}{z} - \frac{z}{x}$

c) $v = \sqrt[x]{e^y}$

Rj. a) Kad vadimo izvod po x-u, samo x tumačimo kao promjenjivu, sve ostalo tumačimo kao broj.

$$\frac{\partial z}{\partial x} = 3x^2 + 5y^2 .$$

Analogno za y-on $\frac{\partial z}{\partial y} = 10xy - 3y^2 .$

b) $$\frac{\partial u}{\partial x} = \frac{1}{y} - z \cdot \left(\frac{1}{x}\right)'_x = \frac{1}{y} - z \cdot (-1) x^{-2} = \frac{1}{y} + \frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = x \cdot (-1) y^{-2} + \frac{1}{z} = -\frac{x}{y^2} + \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = y \cdot \left(\frac{1}{z}\right)'_z - \frac{1}{x} = y \cdot (-1) z^{-2} - \frac{1}{x} = -\frac{y}{z^2} - \frac{1}{x}$$

c) $$\frac{\partial v}{\partial x} = \left(e^{\frac{y}{x}}\right)'_x = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_x = y e^{\frac{y}{x}} \cdot (x^{-1})'_x = -y e^{\frac{y}{x}} \cdot x^{-2} = -\frac{y}{x^2} e^{\frac{y}{x}}$$

$$\frac{\partial v}{\partial y} = \left(e^{\frac{y}{x}}\right)'_y = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_y = \frac{1}{x} e^{\frac{y}{x}}$$

Pronaći vrijednost parcijalnih izvoda datih f-ja u datim tačkama

a) $f(\alpha, \beta) = \cos(m\alpha - n\beta)$, $\alpha = \frac{\pi}{2m}$, $\beta = 0$;

b) $z = \ln(x^2 - y^2)$, $x = 2$, $y = -1$.

Rj.: a) $f'_\alpha = -\sin(m\alpha - n\beta) \cdot (m\alpha - n\beta)'_\alpha = -m \sin(m\alpha - n\beta)$

$f'_\beta = -\sin(m\alpha - n\beta) \cdot (m\alpha - n\beta)'_\beta = n \sin(m\alpha - n\beta)$

$f'_\alpha\left(\frac{\pi}{2m}, 0\right) = -m \sin\frac{\pi}{2} = -m$, $f'_\beta\left(\frac{\pi}{2m}, 0\right) = n \sin\frac{\pi}{2} = n$

b)

$$z'_x = \frac{1}{x^2 - y^2} \cdot 2x$$

$$z'_y = \frac{1}{x^2 - y^2} \cdot (-2y)$$

$$z'_x(2, -1) = \frac{1}{4 - 1} \cdot 2$$

$$z'_y(2, -1) = \frac{1}{4 - 1} \cdot 2 = \frac{2}{3}$$

Nađi sve parcijalne izvode prvog reda f-je

a) $z = x^2y^5 + 3x^3y - 7$ c) $z = (2x^2y^2 - x + 1)^3$ e) $z = \operatorname{arctg}\frac{y}{x}$

b) $z = x^y$ d) $z = \frac{x+y^2}{x^2+y^2+1}$ f) $u = \sqrt{x^2+y^2+z^2}$

g) $u = \ln(x^3 - y^2 + z^4)$

Rj.

a) $z'_x = 2xy^5 + 9x^2y$

$z'_y = x^2 \cdot 5y^4 + 3x^3 = 5x^2y^4 + 3x^3$

b) $z'_x = yx^{y-1}$

$z'_y = x^y \ln x$

c) $z'_x = 3(2x^2y^2 - x + 1)^2(4xy^2 - 1)$

$z'_y = 3(2x^2y^2 - x + 1)^2(4x^2y) = 12x^2y(2x^2y^2 - x + 1)^2$

d) $$z'_x = \frac{1\cdot(x^2+y^2+1) - (x+y^2)\cdot 2x}{(x^2+y^2+1)^2} = \frac{x^2+y^2+1-2x^2-2xy^2}{(x^2+y^2+1)^2} = \frac{-x^2+y^2+1-2xy^2}{(x^2+y^2+1)^2}$$

$$z'_y = \frac{2y(x^2+y^2+1) - (x+y^2)(2y)}{(x^2+y^2+1)^2} = \frac{2x^2y + \cancel{2y^3} + 2y - 2xy - \cancel{2y^3}}{(x^2+y^2+1)^2} = \frac{2y(x^2 - x + 1)}{(x^2+y^2+1)^2}$$

e) $z = \operatorname{arctg}\frac{y}{x}$

$$z'_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{(-1)\cdot y}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2} = \frac{-y}{x^2+y^2}$$

$$z'_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)\cdot x} = \frac{x}{x^2+y^2}$$

f) $u = \sqrt{x^2+y^2+z^2}$, $u'_x = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$

$u'_y = \frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $u'_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$

g) $u = \ln(x^3 - y^2 + z^4)$, $u'_x = \frac{3x^2}{x^3-y^2+z^4}$, $u'_y = \frac{-2y}{x^3-y^2+z^4}$, $u'_z = \frac{4z^3}{x^3-y^2+z^4}$

Proveriti da li f-ja $z = x\ln\frac{y}{x}$ zadovoljava jednakost

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

Rj:

$$\frac{\partial z}{\partial x} = 1\cdot\ln\frac{y}{x} + x\cdot\frac{1}{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_x = \ln\frac{y}{x} + \frac{x^2}{y}\cdot(-1)y(x)^{-2} = \ln\frac{y}{x} - 1$$

F-ju z možemo napisati i u obliku $z = x(\ln y - \ln x)$

$$\frac{\partial z}{\partial y} = x\cdot\frac{1}{y} = \frac{x}{y}$$

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\ln\frac{y}{x} - 1\right) + y\cdot\frac{x}{y} = x\ln\frac{y}{x} - x + x = x\ln\frac{y}{x} = z$$

F-ja $z = x\ln\frac{y}{x}$ zadovoljava datu jednakost.

Ako je $z = x^y\cdot y^x$ dokazati da je

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = z\cdot(x + y + \ln z).$$

Rj:

$$\frac{\partial z}{\partial x} = y x^{y-1}\cdot y^x + x^y\cdot y^x\ln y$$

$$\frac{\partial z}{\partial y} = x^y\ln x\cdot y^x + x^y\cdot x y^{x-1}$$

$$x\cdot\frac{\partial z}{\partial x} = xy\, x^{y-1} y^x + x\ln y\, x^y y^x$$

$$= y\, x^y y^x + x\ln y\, x^y y^x$$

$$y\cdot\frac{\partial z}{\partial y} = y\ln x\cdot x^y y^x + x\cdot x^y y^x$$

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = y x^y y^x + \ln y^x\cdot x^y y^x + x^y y^x \ln x^y + x\, x^y y^x =$$

$$= x^y y^x\left(y + \ln(x^y\cdot y^x) + x\right) = z\cdot(x + y + \ln z)$$

što je i trebalo dobiti

Zadaci za vježbu

Naći parcijalne izvode sljedećih f-ja

1. $z=(5x^3y^3+1)^3$

2. $r=\sqrt{ax^2-by^2}$

3. $v=\ln(x+\sqrt{x^2+y^2})$

4. $\rho=\arcsin\frac{x}{t}$

5. $f(m,n)=(2m)^{3n}$; izračunati f'_m i f'_n u tački $A(\frac{1}{2};2)$

6. $\rho(x,y,z)=\sin^2(3x+2y-z)$; izračunati $\rho'_x(1;-1;1)$, $\rho'_y(1;1;4)$, $\rho'_z(-\frac{1}{2};0;-1)$

7. Provjeriti da li f-ja $v=x^y$ zadovoljava jednakost

$$\frac{x}{y}\cdot\frac{\partial v}{\partial x}+\frac{1}{\ln x}\cdot\frac{\partial v}{\partial y}=2v$$

8. Provjeriti da li f-ja $w=x+\frac{x-y}{y-z}$ zadovoljava jednakost

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}=1.$$

Rješenja:

1. $z'_x=45x^2y^2(5x^3y^2+1)^2$;

$z'_y=30x^3y(5x^3y^2+1)^2$.

2. $\frac{\partial r}{\partial x}=\frac{ax}{r}$; $\frac{\partial r}{\partial y}=-\frac{by}{r}$.

3. $\frac{\partial v}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}$;

$\frac{\partial v}{\partial y}=\frac{y}{(x+\sqrt{x^2+y^2})\sqrt{x^2+y^2}}$.

4. $\frac{\partial \rho}{\partial x}=\frac{|t|}{t\sqrt{t^2-x^2}}$;

$\frac{\partial \rho}{\partial t}=-\frac{x}{|t|\sqrt{t^2-x^2}}$.

5. 12; 0.

6. 0; $2\sin 2$; $-\sin(-1)$

Diferenciranje f-ja više promjenjivih

Posmatrajmo f-ju tri promjenjive $u = f(x, y, z)$. Diferencijal f-je u označavamo sa du i računamo po formuli

$$du = d_x u + d_y u + d_z u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

gdje su $d_x u$, $d_y u$, $d_z u$ parcijalni diferencijali f-je u redom po promjenjivim x, y i z.

$$d_x u = \frac{\partial u}{\partial x} dx, \quad d_y u = \frac{\partial u}{\partial y} dy, \quad d_z u = \frac{\partial u}{\partial z} dz.$$

Odrediti totalne diferencijale f-ja

a) $z = 3x^2y^5$ b) $u = 2x^{yz}$ c) $p = \arccos\frac{1}{uv}$

Rj.

a) Parcijalni izvodi su

$$\frac{\partial z}{\partial x} = 6xy^5, \quad \frac{\partial z}{\partial y} = 15x^2y^4$$

Totalni diferencijal je $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ tj.

$$dz = 6xy^5dx + 15x^2y^4dy$$

b) Parcijalni izvodi su

$$\frac{\partial u}{\partial x} = 2yzx^{yz-1}, \quad \frac{\partial u}{\partial y} = 2x^{yz}\ln x \cdot z, \quad \frac{\partial u}{\partial z} = 2yx^{yz}\ln x$$

Totalni diferencijal je

$$du = 2yzx^{yz-1}dx + 2zx^{yz}\ln x\,dy + 2yx^{yz}\ln x\,dz$$

$$= 2x^{yz}\left(\frac{yz}{x}dx + z\ln x\,dy + y\ln x\,dz\right)$$

c) Parcijalni izvodi su

$$\frac{\partial p}{\partial u} = \frac{-1}{\sqrt{1-\left(\frac{1}{uv}\right)^2}}\cdot\left(\frac{1}{uv}\right)'_u = \frac{-1}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}(-1)(uv)^{-2}\cdot v = \frac{|uv|}{u^2v\sqrt{u^2v^2-1}};$$

$$\frac{\partial p}{\partial v} = \frac{-1}{\sqrt{1-\frac{1}{u^2v^2}}}(-1)(uv)^{-2}\cdot u = \frac{u}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}\cdot\frac{1}{u^2v^2} = \frac{|uv|}{uv^2\sqrt{u^2v^2-1}};$$

Totalni diferencijal

$$dp = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|uv|}{u^2v}du - \frac{|uv|}{uv^2}dv\right) = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|v|\,du}{v\,|u|} - \frac{|u|\,dv}{u\,|v|}\right).$$

Odrediti parcijalne diferencijale f-je $z=\sqrt[3]{x^3+y^3}$.

Rj.

$$z'_x=\frac{\partial z}{\partial x}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3x^2=\frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

$$z'_y=\frac{\partial z}{\partial y}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3y^2=\frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$$

dobijeni izrazi za parcijalne izvode nisu definisani u tački $(0,0)$. Izvode u toj tački treba odrediti po definiciji

$$z'_x(0,0)=\lim_{\varepsilon\to 0}\frac{z(0+\varepsilon,0)-z(0,0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\sqrt[3]{\varepsilon^3+0^3}-0}{\varepsilon}=\lim_{\varepsilon\to 0}1=1$$

$$z'_y(0,0)=\lim_{\varepsilon\to 0}\frac{z(0,0+\varepsilon)-z(0,0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\sqrt[3]{0^3+\varepsilon^3}-0}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon}=1$$

f-ja f ima parcijalne izvode u svim tačkama iz oblasti definisanosti. Parcijalni diferencijali su

$$d_x z=\frac{\partial z}{\partial x}dx=\begin{cases}\dfrac{x^2}{\sqrt[3]{(x^3+y^3)^2}}\,dx, & (x,y)\neq(0,0)\\ dx, & (x,y)=(0,0)\end{cases}$$

$$d_y z=\frac{\partial z}{\partial y}dy=\begin{cases}\dfrac{y^2}{\sqrt[3]{(x^3+y^3)^2}}\,dy, & (x,y)\neq(0,0)\\ dy, & (x,y)=(0,0)\end{cases}$$

(#) Odrediti totalni diferencijal f-je $z=\arcsin\frac{x}{y}$ u tački (4,5).

Rj. F-ja je definisana za $\left|\frac{x}{y}\right|<1$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{x}{y}\right)'_x=\frac{1}{y\sqrt{1-\frac{x^2}{y^2}}}=\frac{1}{\sqrt{y^2-x^2}},\quad \frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)=\frac{-x}{y\sqrt{y^2-x^2}}$$

$$dz=\frac{1}{\sqrt{y^2-x^2}}dx+\frac{-x}{y\sqrt{y^2-x^2}}dy=\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$$

Stavljajući u dobijeni izraz $x=4$ i $y=5$ dobijemo $dz=\frac{1}{15}(5dx-4dy)$

(#) Pomoću totalnog diferencijala približno izračunati $\ln(\sqrt[3]{1{,}03}+\sqrt[4]{0{,}98}-1)$.

Rj. Neka je $z=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)$ gdje je $x=a+\varepsilon=1+0{,}03$ i $y=b+\omega=1-0{,}02$.

Tada je $z(a,b)=\ln(\sqrt{1}-\sqrt{1}-1)=\ln 1=0$; $z=z(a,b)+\Delta z$.

($\Delta z=f(a+\varepsilon,b+\omega)-f(a,b)$ totalni priraštaj f-je u tački (a,b)).

Kako je $\Delta z\approx dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1}\left(\frac{1}{3\sqrt[3]{x^2}}dx+\frac{1}{4\sqrt[4]{y^3}}dy\right)=$

$=\frac{1}{1}\left(\frac{1}{3}\cdot 0{,}03-\frac{1}{4}\cdot 0{,}02\right)=0{,}005$. Pa $z=z_0+\Delta z\approx 0{,}0005$.

(#) Naći totalni diferencijal i totalni priraštaj f-je $z=x^2+y^2+xy$ pri prelazu od tačke (1,1) u tačku (1,1; 0,9).

Rj. po definiciji totalnog priraštaja dobijemo

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=(x+\Delta x)^2+(y+\Delta y)^2+(x+\Delta x)(y+\Delta y)-(x^2+y^2+xy)=$$

$$=x^2+2x\Delta x+\Delta x^2+y^2+2y\Delta y+\Delta y^2+xy+x\Delta y+y\Delta x+\Delta x\Delta y-x^2-y^2-xy$$

$$=2x\Delta x+\Delta x^2+y\Delta x+2y\Delta y+\Delta y^2+x\Delta y+\Delta x\Delta y=(2x+y+\Delta x)\Delta x+(2y+x+\Delta x+\Delta y)\Delta y$$

Ako ubacimo u formulu vrijednosti $x=1$, $y=1$, $\Delta x=1{,}1-1=0{,}1$, $\Delta y=0{,}9-1=-0{,}1$ dobićemo totalni priraštaj date f-je u tački (1,1)

$$\Delta z=(2+1+0{,}1)0{,}1+(2+1+0{,}1-0{,}1)\cdot(-0{,}1)=3{,}1\cdot 0{,}1+3\cdot(-0{,}1)=0{,}31-0{,}3=0{,}01$$

$$dz=(2x+y)dx+(2y+x)dy \qquad dz=(2+1)\cdot 0{,}1+(2+1)\cdot(-0{,}1)=0{,}3-0{,}3=0$$

Diferenciranje složenih f-ja

F-ju z nazivamo složenom f-jom od tri nezavisno promjenjive x, y, t ako je ona zadana putem argumenata $u, v, \ldots, w$:

$$z = F(u, v, \ldots, w)$$

gdje je

$$u = f(x, y, t), \quad v = \varphi(x, y, t), \quad \ldots, \quad w = \psi(x, y, t).$$

Slično bi definisali f-ju od n nezavisno promjenjivih.

Parcijalni izvod složene f-je po jednoj od nezavisnih promjenjivih jednak je sumi proizvoda parcijalnog izvoda f-je po njenom argumentu sa parcijalnim izvodom istog argumenta po nezavisnoj promjenjivoj:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial x};$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial y}; \qquad \ldots (*)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial t}.$$

Ako su svi argumenti $u, v, \ldots, w$ f-je jedne nezavisno promjenjive x, tada je i z složena f-ja po promjenjivoj x. Izvod takve složene f-je (od jedne nezavisno promjenjive) naziva se <u>totalni izvod</u> i dat je preko formule

$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{dw}{dx}. \qquad \ldots (**)$$

(dobije se iz formule totalnog diferencijala f-je $z(u, v, w)$ tako što je podjelimo sa dx).

(#) Naći izvode složenih f-ja

a) $y = u^2 e^v$, $u = \sin x$, $v = \cos x$;

b) $p = u^v$, $u = \ln(x-y)$, $v = e^{\frac{x}{y}}$;

c) $z = x \sin v \cos w$, $v = \ln(x^2+1)$, $w = -\sqrt{1-x^2}$.

Rj: a) Primjetimo da je y složena f-ja po nezavisno promjenjivoj x. Koristimo formulu (**)

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} \cdot \frac{du}{dx} + \frac{\partial y}{\partial v} \cdot \frac{dv}{dx} \overset{(\square)}{=} 2ue^v \cos x - u^2 e^v \sin x$$

$$\frac{\partial y}{\partial u} = 2ue^v, \quad \frac{du}{dx} = \cos x, \quad \frac{\partial y}{\partial v} = u^2 e^v, \quad \frac{dv}{dx} = -\sin x \qquad \ldots(\square)$$

b) p je složena f-ja dvije promjenjive x, y. Koristimo formulu (*)

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial x} = vu^{v-1} \cdot \frac{1}{x-y} + u^v \ln u \cdot \frac{1}{y} e^{\frac{x}{y}}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial y} = vu^{v-1} \cdot \frac{1}{y-x} + u^v \ln u \left(-\frac{x}{y^2} e^{\frac{x}{y}}\right)$$

c) z je složena f-ja jedne promjenjive x. Koristimo formulu (**).

$$\frac{z}{x} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dx}$$

$$\frac{dz}{dx} = \sin v \cos w + x \cos v \cos w \cdot \frac{2x}{x^2+1} - x \sin v \sin w \, \frac{x}{\sqrt{1-x^2}}$$

Naći diferencijal f-je u (naći du) ako je $u=f(\sqrt{x^2+y^2})$.

Rj. $u=f(\sqrt{x^2+y^2})$, uvedimo oznaku $t=\sqrt{x^2+y^2}$.

$$u=f(t)=f(t(x,y)), \qquad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=f_t'\cdot\frac{\partial t}{\partial x}=f_t'\,\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x f_t'}{\sqrt{x^2+y^2}}$$

$$du=\frac{f'_{\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2}\right)(x\,dx+y\,dy)}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}=f_t'\cdot\frac{\partial t}{\partial y}=f_t'\cdot\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y\cdot f_t'}{\sqrt{x^2+y^2}}$$

Ako je $z=\dfrac{y}{f(x^2-y^2)}$ tada je $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}$. Dokazati.

Rj.

$$z=\frac{y}{f(\xi)} \quad \text{gdje je } \xi=x^2-y^2$$

$$\frac{\partial z}{\partial x}=\frac{0\cdot f(\xi)-y\cdot\frac{\partial f}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}}{f^2(\xi)}=\frac{-2xy\frac{\partial f}{\partial\xi}}{f^2(\xi)}$$

$$\frac{\partial z}{\partial y}=\frac{1\cdot f(\xi)-y\cdot\frac{\partial f}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}}{f^2(\xi)}=\frac{f(\xi)+2y^2\frac{\partial f}{\partial\xi}}{f^2(\xi)}$$

(#) Ako je $x^2 = v \cdot w$, $y^2 = u \cdot w$, $z^2 = u \cdot v$ i $f(x,y,z) = F(u,v,w)$

dokazati $x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \cdot \frac{\partial f}{\partial z} = u \cdot \frac{\partial F}{\partial u} + v \cdot \frac{\partial F}{\partial v} + w \cdot \frac{\partial F}{\partial w}$.

Rj: $F(u,v,w) = f(x,y,z) = f(\sqrt{v \cdot w}, \sqrt{u \cdot w}, \sqrt{u \cdot v})$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial u}$$

$$= f'_x \cdot 0 + f'_y \cdot \frac{\sqrt{w}}{2\sqrt{u}} + f'_z \cdot \frac{\sqrt{v}}{2\sqrt{u}} = f'_y \cdot \frac{\sqrt{w}}{2\sqrt{u}} + f'_z \cdot \frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\sqrt{w}}{2\sqrt{v}} + \frac{\partial f}{\partial y} \cdot 0 + \frac{\partial f}{\partial z} \cdot \frac{\sqrt{u}}{2\sqrt{v}} = f'_x \cdot \frac{\sqrt{w}}{2\sqrt{v}} + f'_z \cdot \frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial F}{\partial w} = \frac{\partial f}{\partial x} \cdot \frac{\sqrt{v}}{2\sqrt{w}} + \frac{\partial f}{\partial y} \cdot \frac{\sqrt{u}}{2\sqrt{w}} + \frac{\partial f}{\partial z} \cdot 0 = f'_x \cdot \frac{\sqrt{v}}{2\sqrt{w}} + f'_y \cdot \frac{\sqrt{u}}{2\sqrt{w}}$$

$$u \cdot \frac{\partial F}{\partial u} = \frac{\partial f}{\partial y} \cdot \frac{\sqrt{uw}}{2} + \frac{\partial f}{\partial z} \cdot \frac{\sqrt{uv}}{2}$$

$$v \cdot \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial z} \cdot \frac{\sqrt{uv}}{2}$$

$$w \cdot \frac{\partial F}{\partial w} = \frac{\partial f}{\partial x} \cdot \frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial y} \cdot \frac{\sqrt{uw}}{2}$$

Prema tome $x \cdot \frac{\partial f}{\partial x} + y \cdot \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} = u \frac{\partial F}{\partial u} + v \frac{\partial F}{\partial v} + w \cdot \frac{\partial F}{\partial w}$

q.e.d.

ISPITNI ZADATAK

Ako je $z = z(x,y)$ i $x+y+z = f(x^2+y^2+z^2)$ provjeriti da li je tačna jednakost

$$(y-z)\cdot\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = x-y.$$

Rj. $z = z(x,y) \Rightarrow$ z je f-ja dvije promjenjive x i y.

$$z = f(x^2+y^2+z^2) - x - y$$

$$t = x^2+y^2+z^2$$

$$s = -x-y$$

$$z = f(t) + s$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x} + \frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x} = f'_t\cdot\left(2x + 2z\frac{\partial z}{\partial x}\right) - 1$$

$$\frac{\partial z}{\partial x} - f'_t\cdot 2z\frac{\partial z}{\partial x} = f'_t\cdot 2x - 1$$

$$\frac{\partial z}{\partial x} = \frac{2xf'_t - 1}{1-2zf'_t}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y} + \frac{\partial s}{\partial y}$$

$$\frac{\partial z}{\partial y} = f'_t\cdot\left(2y + 2z\frac{\partial z}{\partial x}\right) - 1$$

$$\frac{\partial z}{\partial y} - 2zf'_t\frac{\partial z}{\partial x} = 2yf'_t - 1$$

$$\frac{\partial z}{\partial y} = \frac{2yf'_t - 1}{1-2zf'_t}$$

$$(y-z)\cdot\frac{\partial z}{\partial x} + (z-x)\frac{\partial z}{\partial y} = \frac{(y-z)(2xf'_t-1)}{1-2zf'_t} + \frac{(z-x)(2yf'_t-1)}{1-2zf'_t} =$$

$$= \frac{\cancel{2xyf'_t} - y - 2xzf'_t + z + 2yzf'_t - z \cancel{- 2xyf'_t} + x}{1-2zf'_t} =$$

$$= \frac{(x-y) - 2xzf'_t + 2yzf'_t}{1-2zf'_t} = \frac{(x-y) + 2zf'_t(-x+y)}{1-2zf'_t} =$$

$$= \frac{(x-y)(1-2zf'_t)}{1-2zf'_t} = x-y$$

(#) Ako je $z = \dfrac{y}{f(x^2-y^2)}$, gdje je f diferencijabilna f-ja,

izračunati $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x} + \dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}$.

Rj.: $z = y f^{-1}(x^2-y^2) = y f^{-1}(u)$, gdje je $u = x^2-y^2$

$$\frac{\partial z}{\partial x} = y(-1) f_u^{-2}(x^2-y^2)\cdot 2x = \frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \left(y f^{-1}(u)\right)_y' = 1\cdot f^{-1}(u) + y\cdot(-1) f_u^{-2}(u)\cdot(-2y) =$$

$$= \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{-2y}{f_u^2(x^2+y^2)} + \frac{1}{y f(x^2-y^2)} + \frac{2y}{f_u^2(x^2+y^2)} =$$

$$= \frac{1}{y f(x^2-y^2)} = \frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)} = \frac{z}{y^2}$$

prema tome $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x} + \dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y} = \dfrac{z}{y^2}$.

(#) Ako je $z = e^y \varphi(y e^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj. $z = e^y \varphi(\xi)$, gdje je $\xi(x,y) = y e^{\frac{x^2}{2y^2}}$

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}} \left(\frac{1}{2}x^2y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2\cdot(-2)y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \varphi}{\partial x} = \frac{x}{y} e^y e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^y \varphi(\xi) + e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} = e^y \varphi(\xi) + e^y e^{\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} -$$
$$- e^y \cdot \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2)\cdot\frac{x}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi} +$$
$$+ xy\left(e^y \varphi(\xi) + e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}\right)$$
$$= \underline{\frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}} \underline{\underline{- yx e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} + xy e^y \varphi(\xi) +$$
$$\underline{\underline{+ xy e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} \underline{- \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}} =$$

$$= xy e^y \varphi(\xi) = xy e^y \varphi(y e^{\frac{x^2}{2y^2}}) = xyz$$

Parcijalni izvodi i diferencijali višeg reda f-ja dvije i više promjenjivih

Parcijalnim izvodima drugog reda f-je $z = f(x,y)$ nazivamo parcijalnim izvodima njenih parcijalnih izvoda prvog reda. Za parcijalne izvode drugog reda upotrebljavamo ove oznake

∂ DELTA

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y) \quad \text{itd.}$$

Analogno se definiraju i označavaju izvodi viših redova.

Diferencijalom drugog reda f-je $z = f(x,y)$ nazivamo diferencijal diferencijala prvog reda te f-je za fiksirane priraste nezavisnih varijabli.

$$d^2 z = d(dz)$$

Analogno se određuju diferencijali f-je z višega nego drugog reda, na primjer $d^3 z = d(d^2 z)$

i općenito $d^n z = d(d^{n-1} z)$ $(n = 2, 3, \dots)$

Ako je $z = f(x,y)$ gdje su x i y nezavisne varijable i f-ja f ima neprekidne parcijalne izvode drugog reda, tada se diferencijal drugog reda f-je z računa po formuli

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2.$$

Općenito, kada postoje neprekidne odgovarajuće derivacije, vrijedi simbolička formula

$$d^n z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n z,$$

koja se formalno razvije po binomnom zakonu.

Naći parcijalne izvode drugog reda f-je

a) $z = e^{-xy}$ c) $u = x^3y + y^3x + z^3y$ e)ˇ $z = \ln tg\frac{x}{y}$

b) $z = x^3 + y^3 - xy$ d)ˇ $u = \ln(x+y-z)$ f)ˇ $u = \sin(x^2+y+z^3)$

Rj: a) $z = e^{-xy}$

$$\frac{\partial z}{\partial x} = e^{-xy}\cdot(-y) = -ye^{-xy}$$

$$\frac{\partial z}{\partial y} = e^{-xy}\cdot(-x) = -xe^{-xy}$$

$$\frac{\partial^2 z}{\partial x^2} = (-y)e^{-xy}\cdot(-y) = y^2 e^{-xy}$$

$$\frac{\partial^2 z}{\partial y^2} = (-x)e^{-xy}\cdot(-x) = x^2 e^{-xy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -e^{-xy} - ye^{-xy}(-x) =$$

$$= e^{-xy}(xy-1)$$

b) $z = x^3 + y^3 - xy$

$$\frac{\partial z}{\partial x} = 3x^2 - y$$

$$\frac{\partial z}{\partial y} = 3y^2 - x$$

$$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial y^2} = 6y, \quad \frac{\partial^2 z}{\partial x \partial y} = -1$$

c) $u = x^3y + y^3x + z^2y$

$$\frac{\partial u}{\partial x} = 3x^2y + y^3$$

$$\frac{\partial u}{\partial y} = x^3 + 3y^2x + z^3$$

$$\frac{\partial u}{\partial z} = 3z^2y$$

$$\frac{\partial^2 u}{\partial x^2} = 6xy, \quad \frac{\partial^2 u}{\partial y^2} = 6xy, \quad \frac{\partial^2 u}{\partial z^2} = 6yz$$

$$\frac{\partial^2 u}{\partial x \partial y} = 3x^2 + 3y^2, \quad \frac{\partial^2 u}{\partial x \partial z} = 0$$

$$\frac{\partial^2 u}{\partial y \partial z} = 3z^2$$

d) $u = \ln(x+y-z)$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y-z} \qquad \frac{\partial u}{\partial z} = \frac{-1}{x+y-z} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{-1}{(x+y-z)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x+y-z} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{-1}{(x+y-z)^2} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{-1}{(x+y-z)^2}$$

završiti sami

...

Proveriti da li vrijedi

a) $u=\ln(x^2+y^2) \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

b) $u=e^{-\alpha x}\cdot\varphi(x-y) \Rightarrow \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\cdot\frac{\partial u}{\partial y}=\alpha^2 u$

Rj. a) $\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial x^2}=\frac{2(x^2+y^2)-2x\cdot(2x)}{(x^2+y^2)^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial y^2}=\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}=0$ što je i trebalo dobiti

b) $u=e^{-\alpha x}\cdot\varphi(x-y)$

$\frac{\partial u}{\partial x}=e^{-\alpha x}\cdot(-\alpha)\varphi(x-y)+e^{-\alpha x}\cdot\varphi'_x=e^{-\alpha x}\left[-\alpha\varphi(x-y)+\varphi'_x\right]$

$\frac{\partial^2 u}{\partial x^2}=e^{-\alpha x}\cdot(-\alpha)\left(-\alpha\varphi(x-y)+\varphi'_x\right)+e^{-\alpha x}\left[-\alpha\varphi'_x+\varphi''_{xx}\right]$

$=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-\alpha\varphi'_x-\alpha\varphi'_x+\varphi''_{xx}\right)=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}\right)$

$\frac{\partial u}{\partial y}=e^{-\alpha x}\cdot\varphi'_y\cdot(-1)=-e^{\alpha x}\varphi'_y$

$\frac{\partial^2 u}{\partial y^2}=-e^{\alpha x}\varphi''_{yy}\cdot(-1)=e^{\alpha x}\cdot\varphi''_{yy}$

$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\frac{\partial u}{\partial y}=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}-\varphi''_{yy}+2\alpha\varphi'_y\right)=$ (u slučaju da je $\varphi'_x=\varphi'_y$ i $\varphi''_{xx}=\varphi''_{yy}$) $=\alpha^2 e^{-\alpha x}\varphi(x-y)=\alpha^2 u$

Naći parcijalne izvode prvog i drugog reda f-je $z=\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\left(x^2+y^2\right)'_x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}\left(x^2+y^2\right)'_y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{2x}{x^2+y^2}\right)'_x = 2\left(\frac{x}{x^2+y^2}\right)'_x = 2\,\frac{1\cdot(x^2+y^2) - x\cdot 2x}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{2x}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2) - x\cdot 2y}{(x^2+y^2)^2} = 2\cdot\frac{x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{(x-y)^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2y}{x^2+y^2}\right)'_y = 2\left(\frac{y}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2}$$

$$= 2\,\frac{x^2-y^2}{(x^2+y^2)^2}$$

Parcijalni izvodi višeg reda složenih f-ja

\# Ako je $u=\varphi(\xi,\eta)$ pričemu je $\xi=x+y$, $\eta=x-y$ izračunati izvode $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Rj.:

$$\frac{\partial u}{\partial x}=\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial \varphi}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial \varphi}{\partial \xi}+\frac{\partial \varphi}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \xi}\right)+\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \eta}\right)=\frac{\partial^2 \varphi}{\partial \xi^2}\frac{\partial \xi}{\partial x}+\frac{\partial^2 \varphi}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}+$$

$$+\frac{\partial^2 \varphi}{\partial \eta \partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial^2 \varphi}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial^2 \varphi}{\partial \xi^2}+2\frac{\partial^2 \varphi}{\partial \xi \partial \eta}+\frac{\partial^2 \varphi}{\partial \eta^2}$$

$$=\left(\frac{\partial \varphi}{\partial \xi}+\frac{\partial \varphi}{\partial \eta}\right)^2$$

$$\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \xi}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \eta}\right)=\frac{\partial^2 \varphi}{\partial \xi^2}\frac{\partial \xi}{\partial y}+\frac{\partial^2 \varphi}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial y}+$$

$$+\frac{\partial^2 \varphi}{\partial \eta \partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial^2 \varphi}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y}=\frac{\partial^2 \varphi}{\partial \xi^2}-\frac{\partial^2 \varphi}{\partial \eta^2}$$

(#) Ako je $u = \frac{\varphi(x-y) + \psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}(\varphi(x-y) + \psi(x+y)) = x^{-1}(\varphi(x-y) + \psi(x+y))$$

$$u'_x = \frac{\partial u}{\partial x} = (-1)x^{-2}(\varphi(\underbrace{x-y}_{s}) + \psi(\underbrace{x+y}_{t})) + \frac{1}{x}\left(\varphi'_s \cdot s'_x + \psi'_t \cdot t'_x\right) =$$

$$= \frac{-1}{x^2}\left[\varphi(x-y) + \psi(x+y)\right] + \frac{1}{x}\left(\varphi'_s \cdot 1 + \psi'_t \cdot 1\right)$$

$$x^2 \frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x(\varphi'_s + \psi'_t)$$ gdje su $s = x-y$ i $t = x+y$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = \underline{-\varphi'_s \cdot 1} \underline{\underline{- \psi'_t \cdot 1}} \underline{+ 1 \cdot (\varphi'_s} \underline{\underline{+ \psi'_t}}) + x(\varphi''_{ss} \cdot 1 + \psi''_{tt} \cdot 1)$$

$$= x(\varphi''_{ss} + \psi''_{tt}) \quad \ldots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}(\varphi'_s \cdot s'_y + \psi'_t \cdot t'_y) = \frac{1}{x}(\varphi'_s \cdot (-1) + \psi'_t \cdot 1) = \frac{1}{x}(-\varphi'_s + \psi'_t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}(-\varphi''_{ss} \cdot s'_y + \psi''_{tt} \cdot t'_y) = \frac{1}{x}(\varphi''_{ss} + \psi''_{tt})$$

$$x^2 \frac{\partial^2 u}{\partial y^2} = x(\varphi''_{ss} + \psi''_{tt}) \quad \ldots (2)$$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2} \overset{(1) i (2)}{=\!=} 0$$ traženo rješenje

Zadaci za vježbu

§ 3. Izvodi i diferencijali funkcija više promenljivih

Parcijalni izvodi

3032. Zapremina gasa v je funkcija njegove temperature i pritiska: $v=f(p, T)$. Kad pritisak gasa ostaje konstantan, srednjim koeficijentom širenja gasa pri promeni njegove temperature od T_1 do T_2 naziva se veličina $\frac{v_2-v_1}{v(T_2-T_1)}$. Šta treba zvati koeficijentom širenja gasa pri konstantnom pritisku za datu temperaturu T_0?

3033. Temperatura θ u datoj tački A štapa Ox je funkcija apscise x tačke A i vremena t: $\theta=f(x, t)$. Kakav fizički smisao imaju parcijalni izvodi $\frac{\partial\theta}{\partial t}$ i $\frac{\partial\theta}{\partial x}$?

3034. Površina S pravougaonika čija je osnovica b i visina h izražava se obrascem $S=bh$. Naći $\frac{\partial S}{\partial h}$ i $\frac{\partial S}{\partial x}$ i objasniti geometrijski smisao rezultata.

3035. Date su dve funkcije: $u=\sqrt{a^2-x^2}$ (a je konstanta) i $z=\sqrt{y^2-x^2}$. Naći $\frac{du}{dx}$ i $\frac{\partial z}{\partial x}$ i uporediti rezultate.

U zadacima 3036—3084 naći parcijalne izvode datih funkcija po svakoj od nezavisno promenljivih ($x, y, z, u, v, t, \varphi$ i ψ su promenljive veličine).

3036. $z=x-y$.

3037. $z=x^3y-y^3x$.

3038. $\theta=axe^{-t}+bt$ (a, b su konstante).

3039. $z=\frac{u}{v}+\frac{v}{u}$.

3040. $z=\frac{x^3+y^3}{x^2+y^2}$.

3041. $z=(5x^2y-y^3+7)^3$.

3042. $z=x\sqrt{y}+\frac{y}{\sqrt[3]{x}}$.

3043. $z=\ln(x+\sqrt{x^2+y^2})$.

3044. $z=\operatorname{arctg}\frac{x}{y}$.

3045. $z=\frac{1}{\operatorname{arctg}\frac{y}{x}}$.

3046. $z=x^y$.

3047. $z=\ln(x^2+y^2)$.

3048. $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

3049. $z=\arcsin\frac{\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}}$.

3050. $z=\ln\operatorname{tg}\frac{x}{y}$.

3051. $z=e^{-\frac{x}{y}}$.

3052. $z=\ln(x+\ln y)$.

3053. $u=\operatorname{arctg}\frac{v+w}{v-w}$.

3054. $z=\sin\frac{x}{y}\cos\frac{y}{x}$.

3055. $z=\left(\frac{1}{3}\right)^{\frac{y}{x}}$.

3056. $z=(1+xy)^y$.

3057. $z=xy\ln(x+y)$.

3058. $z=x^{xy}$.

3059. $u=xyz$.

3060. $u=xy+yz+zx$.

3061. $u=\sqrt{x^2+y^2+z^2}$.

3062. $u=x^3+yz^2+3\,yx-x+z$.

3063. $w=xyz+yzv+zvk+vxy$.

3064. $u=e^{x(x^2+y^2+z^2)}$.

3066. $u=\ln(x+y+z)$

3065. $u=\sin(x^2+y^2+z^2)$.

3075. $z=\operatorname{arctg}\sqrt{x^y}$.

3067. $u=x^{\frac{y}{z}}$.

3068. $u=x^{yz}$.

3069. $f(x,\ y)=x+y-\sqrt{x^2+y^2}$ u tački (3, 4).

3070. $z=\ln\left(x+\dfrac{y}{2x}\right)$ u tački (1, 2).

3071. $z=(2x+y)^{2x+y}$.

3072. $z=(1+\log_y x)^3$.

3073. $z=xye^{\sin\pi xy}$.

3074. $z=(x^2+y^2)\dfrac{1-\sqrt{x^2+y^2}}{1+\sqrt{x^2-y^2}}$.

3076. $z=2\sqrt{\dfrac{1-\sqrt{xy}}{1+\sqrt{xy}}}$.

3077. $z=\ln[xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}]$.

3078. $z=\sqrt{1-\left(\dfrac{x+y}{xy}\right)^2}+\arcsin\dfrac{x+y}{xy}$.

3079. $z=\operatorname{arctg}\left(\operatorname{arctg}\dfrac{y}{x}\right)-\dfrac{1}{2}\dfrac{\operatorname{arctg}\dfrac{x}{y}-1}{\operatorname{arctg}\dfrac{x}{y}+1}-\operatorname{arctg}\dfrac{x}{y}$.

3080. $u=\dfrac{k}{(x^2+y^2+z^2)^2}$.

3081. $u=\operatorname{arctg}(x-y)^z$.

3082. $u=(\sin x)^{yz}$.

3083. $u=\ln\dfrac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}$.

3084. $w=\dfrac{1}{2}\operatorname{tg}^2(x^2y^2+z^2v^2-xyzv)+\ln\cos(x^2y^2+z^2v^2-xyzv)$.

3085. $n=\dfrac{\cos(\varphi-2\psi)}{\operatorname{sos}(\varphi+2\psi)}$. Naći $\left(\dfrac{\partial u}{\partial\psi}\right)_{\substack{\varphi=\frac{\pi}{4}\\ \psi=\pi}}$

3086. $u=\sqrt{az^3-bt^3}$. Naći $\dfrac{\partial u}{\partial z}$ i $\dfrac{\partial u}{\partial t}$ za $z=b,\ t=a$.

3087. $z=\dfrac{x\cos y-y\cos x}{1+\sin x+\sin y}$. Naći $\dfrac{\partial z}{\partial x}$ i $\dfrac{\partial z}{\partial y}$ za $x=y=0$.

3088. $u=\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}$. Naći $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{x=0\\ y=0\\ z=\frac{\pi}{4}}}$

3089. $u=\ln(1+x+y^2+z^3)$. Naći $u'_x+u'_y+u'_z$ za $x=y=z=1$.

3090. $f(x,\ y)=x^3y-y^3x$. Naći $\left(\dfrac{\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial f}{\partial y}}\right)_{\substack{x=1\\ y=2}}$

3091. Koliki ugao zaklapa tangenta u tački (2, 4, 5) krive $\begin{cases} z=\dfrac{x^2+y^2}{4} \\ y=4 \end{cases}$ sa pozitivnim pravcem apscisne ose.

3092. Koliki ugao zaklapa tangenta krive $\begin{cases} z=\sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ u tački $(1, 1, \sqrt{3})$ sa pozitivnim pravcem ordinatne ose.

3093. Pod kojim se uglom seku ravne krive po kojima ravan $y=2$ preseca površine $z=x^2+\dfrac{y^2}{6}$ i $z=\dfrac{x^2+y^2}{3}$?

Diferencijali. Približna računanja

U zadacima 3094—3097 naći parcijalne diferencijale datih funkcija po svakoj od nezavisno promenljivih.

3094. $z=xy^3-3x^2y^2+2y^4$.

3095. $z=\sqrt{x^2+y^2}$.

3096. $z=\dfrac{xy}{x^2+y^2}$.

3097. $u=\ln(x^3+2y^3-z^3)$.

3098. $z=\sqrt[3]{x+y^2}$. Naći $d_y z$ za $x=2,\ y=5,\ \Delta y=0{,}01$.

3099. $z=\sqrt{\ln xy}$. Naći $d_x z$ za $x=1,\ y=1,2,\ \Delta x=0{,}016$.

3100. $u=p-\dfrac{qr}{p}+\sqrt{p+q+r}$. Naći $d_p u$ za $p=1,\ q=3,\ r=5,\ \Delta p=0{,}01$.

U zadacima 3101—3109 naći totalne diferencijale datih funkcija

3101. $z=x^2y^4-x^3y^3+x^4y^3$.

3102. $z=\dfrac{1}{2}\ln(x^2+y^2)$.

3103. $z=\dfrac{x+y}{x-y}$.

3104. $z=\arcsin\dfrac{x}{y}$.

3105. $z=\sin(xy)$.

3106. $z=\operatorname{arctg}\dfrac{x+y}{1-xy}$.

3107. $z=\dfrac{x^2+y^2}{x^2-y^2}$.

3108. $z=\operatorname{arctg}(xy)$.

3109. $u=x^{yz}$.

§ 4. Diferenciranje funkcija

Posredna funkcija

3124. $u=e^{x-2y}$, pri čemu je $x=\sin t$, $y=t^3$; $\frac{du}{dt}=?$

3125. $u=z^2+y^2+zy$, $z=\sin t$, $y=e^t$; $\frac{du}{dt}=?$

3126. $z=\arcsin(x-y)$, $x=3t$, $y=4t^3$; $\frac{dz}{dt}=?$

3127. $z=x^2y-y^2x$, gde je $x=u\cos v$, $y=u\sin v$; $\frac{\partial z}{\partial u}=?$ $\frac{\partial z}{\partial v}=?$

3128. $z=x^2\ln y$, $x=\frac{u}{v}$, $y=3u-2v\cdot$ $\frac{\partial z}{du}=?$ $\frac{\partial z}{\partial v}=?$

3129. $u=\ln(e^x-e^y)$; $\frac{\partial u}{\partial x}=?$ Naći $\frac{du}{dx}$, Ako je $y=x^3$.

3130. $z=\operatorname{arctg}(xy)$; naći $\frac{dz}{dx}$, ako je $y=e^x$.

3131. $u=\arcsin\frac{x}{z}$, gde je $z=\sqrt{x^2+1}$; $\frac{du}{dx}=?$

3132. $z=\operatorname{tg}(3t+2x^2-y)$, $x=\frac{1}{t}$, $y=\sqrt{t}$; $\frac{dz}{dt}=?$

3133. $u=\frac{e^{ax}(x-z)}{a^2+1}$, $y=a\sin x$, $z=\cos x$; $\frac{du}{dx}=?$

3134. $z=\frac{xy\operatorname{arctg}(xy+x+y)}{x+y}$; $dz=?$

3135. $z=(x^2+y^2)e^{\frac{x^2+y^2}{xy}}$; $\frac{\partial z}{\partial x}=?$ $\frac{\partial z}{\partial y}=?$ $dz=?$

3136. $z=f(x^2-y^2, e^{xy})$; $\frac{\partial z}{\partial x}=?$ $\frac{\partial z}{\partial y}=?$

3137. Uveriti se da funkcija $z=\operatorname{arctg}\frac{x}{y}$, u kojoj je $x=u+v$, $y=u-v$, zadovoljava relaciju

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{v^2+u^2}.$$

3138. Uveriti se da funkcija $z=\varphi(x^2+y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava relaciju:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

3139. $u=\sin x+F(\sin y-\sin x)$; uveriti se da je $\frac{\partial u}{\partial y}\cos x+\frac{\partial u}{\partial x}\cos y=$ $=\cos x\cos y$, ma kakva bila diferencijabilna funkcija F.

3140. $z=\frac{y}{f(x^2-y^2)}$, uveriti se da je $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{y}{y^2}$, ma kakva bila diferencijabilna funkcija f.

3141. Pokazati da homogena diferencijabilna funkcija $z=F\left(\frac{y}{x}\right)$ nultog stepena homogenosti (vidi zad. 2961) zadovoljava relaciju $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

3142. Pokazati da homogena funkcija $u=x^k F\left(\frac{z}{x};\ \frac{y}{x}\right)$, k-tog stepena homogenosti, u kojoj je F diferencijabilna funkcija, zadovoljava relaciju

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=ku.$$

3143. Proveriti tvrđenje formulisano u zadatku 3142 na funkciji

$$u=x^5\sin\frac{z^2+y^2}{x^2}.$$

3144. Neka je funkcija $f(x, y)$ diferencijabilna. Dokazati da, ako se promenljive x i y zamene linearnim homogenim funkcijama promenljivih X i Y, onda je tako dobijena funkcija $F(X, Y)$ vezana sa funkcijom $f(x, y)$ sledećom relacijom:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}.$$

§ 5. Izvodi višeg reda

3181. $z=x^3+xy^2-5xy^3+y^5$. Uveriti se da je: $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

3182. $z=x^y$. Uveriti se da je $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

3183. $z=e^x(\cos y+x\sin y)$. Uveriti se da je

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}.$$

3184. $z=\operatorname{arctg}\frac{y}{x}$. Uveriti se da je $\frac{\partial^3 z}{\partial y^2\partial x}=\frac{\partial^3 z}{\partial x\partial y^2}$.

U zadacima 3185—3192 naći $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\partial y}$, i $\frac{\partial^2 z}{\partial y^2}$ za date frnkcije.

3185. $z=\frac{1}{3}\sqrt{(x^2+y^2)^3}$.

3186. $z=\ln(x+\sqrt{x^2+y^2})$.

3187. $z=\operatorname{arctg}\frac{x+y}{1-xy}$.

3188. $z=\sin^2(ax+by)$.

3189. $z=e^{xe^y}$.

3190. $z=\frac{x-y}{x+y}$.

3191. $z=y^{\ln x}$.

3192. $z=\arcsin(xy)$.

3193. $u=\sqrt{x^2+y^2+z^2-2xz}$; $\frac{\partial^2 u}{\partial y\partial z}=?$

3194. $z=e^{xy^2}$; $\frac{\partial^3 z}{\partial x^2\partial y}=?$

3195. $s=\ln(x^2+y^2)$; $\frac{\partial^3 z}{\partial x\partial y^2}=?$

3196. $z=\sin xy$; $\frac{\partial^3 z}{\partial x\partial y^2}=?$

3197. $w=e^{xyz}$; $\frac{\partial^3 w}{\partial x\partial y\partial z}=?$

3198. $v=x^m y^n z^p$; $\frac{\partial^6 v}{\partial x\partial y^3\partial z^2}=?$

3199. $z=\ln(e^x+e^y)$; uveriti se da je $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=1$ i da je

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0.$$

3200. $u=e^x(x\cos y-y\sin y)$. Pokazati da je $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

3201. $u=\ln\dfrac{1}{\sqrt{x^2+y^2}}$; pokazati da je $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0.$

3202. $u=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$; pokazati da je $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=0.$

3203. $r=\sqrt{x^2+y^2+z^2}$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{2}{r},\quad \frac{\partial^2(\ln r)}{\partial x^2}+\frac{\partial^2(\ln r)}{\partial y^2}+\frac{\partial^2(\ln r)}{\partial z^2}=\frac{1}{r^2}.$$

3204. Za koje vrednosti konstante a funkcija $v=x^3+axy^2$ zadovoljava jednačinu

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0?$$

3205. $z=\dfrac{y}{y^2-a^2x^2}$; pokazati da je $\dfrac{\partial^2 z}{\partial x^2}=a^2\dfrac{\partial^2 z}{\partial y^2}.$

3206. $v=\dfrac{1}{x-y}+\dfrac{1}{y-z}+\dfrac{1}{z-x}$; uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+2\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 v}{\partial z\,\partial x}\right)=0.$$

3207. $z=f(x,y)$, $\xi=x+y$, $\eta=x-y$; uveriti se da je

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial\xi\,\partial\eta}.$$

3208. $v=x\ln(x+r)-r$, gde je $r^2=x^2+y^2$. Uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{1}{x+r}.$$

3209. Izvesti obrazac za drugi izvod $\dfrac{d^2y}{dx^2}$ funkcije y, definisane implicitno jednačinom $f(x,y)=0$.

3210. $y=\varphi(x-at)+\psi(x+at)$. Pokazati da je

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2},$$

ma kakve bile dvaput diferencijabilne funkcije φ i ψ.

3211. $u=\varphi(x)+\psi(y)+(x-y)\psi'(y)$. Uveriti se da je

$$(x-y)\frac{\partial^2 y}{\partial x\,\partial y}=\frac{\partial u}{\partial y}$$

(φ i ψ su dvaput diferencijabilne funkcije).

3212. $z=y\varphi(x^2-y^2)$. Uveriti se da je

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

(φ je diferencijabilna funkcija).

3213. $r = x\varphi(x+y) + y\psi(x+y)$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2} - 2\frac{\partial^2 r}{\partial x \partial y} + \frac{\partial^2 r}{\partial y^2} = 0$$

(φ i ψ su dvaput diferencijabilne funkcije).

3214. $u = \frac{1}{y}[\varphi(ax+y) + \psi(ax-y)]$. Pokazaii da je

$$\frac{\partial^2 u}{\partial x^2} = \frac{a^2}{y^2} \cdot \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right).$$

3215. $u = \frac{1}{x}[\varphi(x-y) + \psi(x+y)]$. Pokazati da je

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = x^2 \frac{\partial^2 u}{\partial y^2}.$$

3216. $u = xe^y + ye^x$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 y}{\partial y^3} = x\frac{\partial^3 u}{\partial x \partial y^2} + y\frac{\partial^3 u}{\partial x^2 \partial y}.$$

3217. $u = e^{xyz}$. Pokazati da je

$$\frac{\partial^3 y}{\partial x \partial y \partial z} = xy\frac{\partial^2 u}{\partial x \partial y} + 2x\frac{\partial u}{\partial x} + u.$$

3218. $u = \ln\frac{x^2-y^2}{xy}$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 u}{\partial y^3} = 2\left(\frac{1}{y^3} - \frac{1}{x^3}\right).$$

U zadacima 3219—3224 naći diferencijale drugog reda za date funkcije.

3219. $z = xy^2 - x^2 y$.

3220. $z = \ln(x-y)$.

3221. $z = \frac{1}{2(x^2+y^2)}$.

3222. $z = x\sin^2 y$.

3223. $z = e^{xy}$.

3224. $u = xyz$.

3225. $z = \sin(2x+y)$. Naći $d^3 z$ u tačkama $(0, \pi)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3226. $u + \sin(x+y+z)$; $d^2 u = ?$

3227. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; $d^2 z = ?$

3228. $z^3 - 3xyz = a^3$; $d^2 z = ?$

3229. $3x^2y^2 + 2z^2xy - 2zx^3 + 4zy^3 - 4 = 0$. Naći $d^2 z$ u tački $(2, 1, 2)$.

Rješenja

3032. $\frac{1}{v}\frac{\partial v}{\partial T}$ za $T = T_0$.

3033. $\frac{\partial \theta}{\partial t}$ — brzina menjanja temperature u datoj tački; $\frac{\partial \theta}{\partial x}$ — brzina menjanja temperature u odnosu na dužinu (duž štapa), u datom trenutku vremena.

3034. $\frac{\partial S}{\partial h} = b$ — brzina menjanja površine u zavisnosti od visine pravougaonika; $\frac{\partial S}{\partial h} = h$ — brzina menjanja površine u zavisnosti od osnovice pravougaonika.

3036. $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = -1$.

3037. $\frac{\partial z}{\partial x} = 3x^2y - y^3$; $\frac{\partial z}{\partial y} = x^3 - 3y^2x$.

3038. $\frac{\partial \theta}{\partial x} = ae^{-t}$; $\frac{\partial \theta}{\partial t} = -axe^{-t} + b$.

3039. $\frac{\partial z}{\partial u} = \frac{1}{v} - \frac{v}{u^2}$; $\frac{\partial z}{\partial v} = -\frac{u}{v^2} + \frac{1}{u}$.

3040. $\frac{\partial z}{\partial x} = \frac{x^4 + 3x^2y^2 - 2xy^3}{(x^2+y^2)^2}$; $\frac{\partial z}{\partial y} = \frac{y^4 + 3x^2y^2 - 2x^3y}{(x^2+y^2)^2}$.

3041. $\frac{\partial z}{\partial x} = 30xy\,(5x^2y - y^3 + 7)^2$; $\frac{\partial z}{\partial y} = 3\,(5x^2y - y^3 + 7)^2\,(5x^2 - 3y^2)$.

3042. $\frac{\partial z}{\partial x} = \sqrt{y} - \frac{y}{3\sqrt[3]{x^4}}$; $\frac{\partial z}{\partial y} = \frac{x}{2\sqrt{y}} + \frac{1}{\sqrt[3]{x}}$.

3043. $\frac{\partial z}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}$; $\frac{\partial z}{\partial y} = \frac{y}{x^2 + y^2 + x\sqrt{x^2+y^2}}$.

3044. $\frac{\partial z}{\partial x} = \frac{y}{x^2+y^2}$; $\frac{\partial z}{\partial y} = -\frac{x}{x^2+y^2}$.

3045. $\frac{\partial z}{\partial x} = \frac{y}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}$; $\frac{\partial z}{\partial y} = -\frac{x}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}$.

3046. $\frac{\partial z}{\partial x} = yx^{y-1}$; $\frac{\partial z}{\partial y} = x^y \ln x$.

3047. $\frac{\partial z}{\partial x} = \frac{2x}{x^2+y^2}$; $\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}$.

3048. $\frac{\partial z}{\partial x} = -\frac{2}{\sqrt{x^2+y^2}}$; $\frac{\partial z}{\partial y} = \frac{2x}{y\sqrt{x^2+y^2}}$.

3049. $\frac{\partial z}{\partial x} = \frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}$; $\frac{\partial z}{\partial y} = -\frac{x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}$.

3050. $\frac{\partial z}{\partial x} = \frac{2}{y\sin\frac{2x}{y}}$; $\frac{\partial z}{\partial y} = -\frac{2x}{y^2\sin\frac{2x}{y}}$.

3051. $\frac{\partial z}{\partial x} = -\frac{1}{y}e^{-\frac{x}{y}}$; $\frac{\partial z}{\partial y} = \frac{x}{y^2}e^{-\frac{x}{y}}$.

3052. $\frac{\partial z}{\partial x}=\frac{1}{x+\ln y}$; $\quad \frac{\partial z}{\partial y}=\frac{1}{y(x+\ln y)}$.

3053. $\frac{\partial u}{\partial v}=-\frac{w}{v^2+w^2}$; $\quad \frac{\partial u}{\partial w}=\frac{v}{v^2+w^2}$.

3054. $\frac{\partial z}{\partial x}=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\sin\frac{y}{x}$;

$\frac{\partial z}{\partial y}=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}$.

3055. $\frac{\partial z}{\partial x}=\frac{y}{x^2}3^{-\frac{y}{x}}\ln 3$; $\quad \frac{\partial z}{\partial y}=-\frac{1}{x}3^{-\frac{y}{x}}\ln 3$.

3056. $\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}$; $\quad \frac{\partial z}{\partial y}=xy(1+xy)^{y-1}+(1+xy)^y\ln(1+xy)$.

3057. $\frac{\partial z}{\partial x}=y\ln(x+y)+\frac{xy}{x+y}$; $\quad \frac{\partial z}{\partial y}=x\ln(x+y)+\frac{xy}{x+y}$.

3058. $\frac{\partial z}{\partial x}=x^{x^y}x^{y-1}(y\ln x+1)$; $\quad \frac{\partial z}{\partial y}=x^y x^{x^y}\ln^2 x$

3059. $\frac{\partial u}{\partial x}=yz$; $\quad \frac{\partial u}{\partial y}=xz$; $\quad \frac{\partial u}{\partial z}=xy$.

3060. $\frac{\partial u}{\partial x}=y+z$; $\quad \frac{\partial u}{\partial y}=x+z$; $\quad \frac{\partial u}{\partial z}=x+y$.

3061. $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$; $\quad \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}$; $\quad \frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$.

3062. $\frac{\partial u}{\partial x}+3x^2+3y-1$; $\quad \frac{\partial u}{\partial y}=z^2+3x$; $\quad \frac{\partial u}{\partial z}=2yz+1$.

3063. $\frac{\partial w}{\partial x}=yz+vz+vy$; $\quad \frac{\partial w}{\partial y}=xz+zv+vx$; $\quad \frac{\partial w}{\partial z}=xy+yv+vx$; $\quad \frac{\partial w}{\partial v}=yz+xz+xy$.

3064. $\frac{\partial u}{\partial x}=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$;

$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)}$; $\quad \frac{\partial u}{\partial z}=2xze^{x(x^2+y^2+z^2)}$.

3065. $\frac{\partial u}{\partial x}=2x\cos(x^2+y^2+z^2)$; $\quad \frac{\partial u}{\partial y}=2y\cos(x^2+y^2+z^2)$;

$\frac{\partial u}{\partial z}=2z\cos(x^2+y^2+z^2)$.

3066. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{1}{x+y+z}$.

3067. $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}$; $\quad \frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x$; $\quad \frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$.

3068. $\frac{\partial u}{\partial x}=y^z x^{y^z-1}$; $\quad \frac{\partial u}{\partial y}=zy^{z-1}x^{y^z}\ln x$; $\quad \frac{\partial u}{\partial z}=y^z x^{y^z}\ln x\ln y$.

3069. $\frac{2}{5}$, $\frac{1}{5}$. 3070. 0, $\frac{1}{4}$.

3071. $\frac{\partial z}{\partial x}=2(2x+y)^{2x+y}[1+\ln(2x+y)]$;

$\frac{\partial z}{\partial y}=(2x+y)^{2x+y}[1+\ln(2x+y)]$.

3072. $\frac{\partial z}{\partial x}=\frac{3}{x\ln y}\left(1+\frac{\ln x}{\ln y}\right)^2$; $\frac{\partial z}{\partial y}=-\frac{3\ln x}{y\ln^2 y}\left(1+\frac{\ln x}{\ln y}\right)^2$.

3073. $\frac{\partial z}{\partial x}=ye^{\sin\pi xy}(1+\pi xy\cos\pi xy)$;

$\frac{\partial z}{\partial y}=xe^{\sin\pi xy}(1+\pi xy\cos\pi xy)$.

3074. $\frac{\partial z}{\partial x}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2x$;

$\frac{\partial z}{\partial y}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2y$.

3075. $\frac{\partial z}{\partial x}=\frac{y\sqrt{x^y}}{2x(1+x^y)}$; $\frac{\partial z}{\partial y}=\frac{\sqrt{x^y}\ln x}{2(1+x^y)}$.

3076. $\frac{\partial z}{\partial x}=-\frac{y}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$; $\frac{\partial z}{\partial y}=-\frac{x}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$.

3077. $\frac{\partial z}{\partial x}=\frac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$; $\frac{\partial z}{\partial y}=\frac{x^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$.

3078. $\frac{\partial z}{\partial x}=-\frac{1}{x^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$; $\frac{\partial z}{\partial y}=-\frac{1}{y^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$.

3079. $$\frac{\partial z}{\partial x}=\frac{y\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2};$$

$$\frac{\partial z}{\partial y}=-\frac{x\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2}.$$

3080. $\frac{\partial u}{\partial x}=-\frac{4kx}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial y}=-\frac{4ky}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial z}=-\frac{4kz}{(x^2+y^2+z^2)^3}$.

3081. $\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

3082. $\frac{\partial u}{\partial x}=yz(\sin x)^{yz-1}\cos x$; $\frac{\partial u}{\partial y}=z(\sin x)^{yz}\ln\sin x$;

$\frac{\partial u}{\partial z}=y(\sin x)^{yz}\ln\sin x$.

3083. $\frac{\frac{\partial u}{\partial x}}{x}=\frac{\frac{\partial u}{\partial y}}{y}=\frac{\frac{\partial u}{\partial z}}{z}=\frac{2}{r(r^2-1)}$, gde $r=\sqrt{x^2+y^2+z^2}$.

3084. $\frac{\partial w}{\partial x}=(2xy^2-yzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial y}=(2x^2y-xzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial z}=(2zv^2-xyv)\operatorname{tg}^3\alpha$;

$\frac{\partial w}{\partial v}=(2z^2v-xyz)\operatorname{tg}^3\alpha$, gde je $\alpha=x^2y^2+z^2v^2-xyzv$.

3085. 4. 3086. $\left(\frac{\partial u}{\partial z}\right)_{\substack{z=b\\t=a}}=\frac{3b}{2}\sqrt{\frac{ab}{b^2-a^2}}$;

$$\left(\frac{\partial u}{\partial t}\right)_{\substack{z=b\\t=a}}=-\frac{3a}{3}\sqrt{\frac{ab}{b^2-a^2}};$$

3087. 1 і -1. 3088. $\frac{\sqrt{2}}{2}$. 3089. $\frac{3}{2}$. 3090. $\frac{13}{22}$. 3091. 45°.

3092. 30°. 3093. $\operatorname{arctg}\frac{4}{7}$.

3094. $d_x z=(y^3-6xy^2)\,dx$; $d_y z=(3xy^2-6x^2y+8y^3)\,dy$.

3095. $d_x z=\frac{x\,dx}{\sqrt{x^2+y^2}}$; $d_y z=\frac{y\,dy}{\sqrt{x^2+y^2}}$.

3096. $d_x z=\frac{y(y^2-x^2)\,dx}{(x^2+y^2)^2}$; $d_y z=\frac{x(x^2-y^2)\,dy}{(x^2+y^2)^2}$.

3097. $d_x u=\frac{3x^2\,dx}{x^3+2y^3-z^3}$; $d_y u=\frac{6y^3\,dy}{x^3+2y^3-z^3}$; $d_z u=\frac{-3z^2\,dz}{x^3+2y^3-z^3}$.

3098. $\frac{1}{270}$. 3099. $\approx 0{,}0187$. 3100. $\frac{97}{600}$.

3101. $xy[(2y^3-3xy^2+4x^2y)\,dx+(4y^2x-3yx^2+2x^3)\,dy]$.

3102. $\frac{x\,dx+y\,dy}{x^2+y^2}$. 3103. $\frac{2(x\,dy-y\,dx)}{(x-y)^2}$. 3104. $\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$

3105. $(x\,dy+y\,dx)\cos(xy)$. 3106. $\frac{dx}{1+x^2}+\frac{dy}{1+y^2}$.

3107. $\frac{4xy(x\,dy-y\,dx)}{(x^2-y^2)^2}$. 3108. $\frac{x\,dy+y\,dx}{1+x^2y^2}$.

3109. $x^{xy-1}(yz\,dx+zx\ln x\,dx+xy\ln x\,dz)$,

3124. $e^{\sin t-2t^3}(\cos t-6t^2)$. 3125. $\sin 2t+2e^{2t}+e^t(\sin t+\cos t)$.

3126. $\dfrac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$. 3127. $\dfrac{\partial z}{\partial u}=3u^2\sin v\cos v(\cos v-\sin v)$;

$$\frac{\partial z}{\partial v}=u^3(\sin v+\cos v)(1-3\sin v\cos v).$$

3128. $\dfrac{\partial z}{\partial u}=2\dfrac{u}{v^2}\ln(3u-2v)+\dfrac{3u^2}{v^2(3u-2v)}$;

$$\frac{\partial z}{\partial v}=-\frac{2u^2}{v^3}\ln(3u-2v)-\frac{2u^2}{v^2(3u-2v)}.$$

3129. $\dfrac{\partial u}{\partial x}=\dfrac{e^x}{e^x+e^y}$; $\dfrac{du}{dx}=\dfrac{e^x+3e^{x^3}x^2}{e^x+e^{x^3}}$.

3130. $\dfrac{dz}{dx}=\dfrac{e^x(x+1)}{1+x^2e^{2x}}$. 3131. $\dfrac{du}{dx}=\dfrac{1}{1+x^2}$.

3132. $\dfrac{dz}{dt}=\left(3-\dfrac{4}{t^3}-\dfrac{1}{2\sqrt{t}}\right)\sec^2\left(3t+\dfrac{2}{t^2}-\sqrt{t}\right)$.

3133. $\dfrac{du}{dx}=e^{ax}\sin x$. 3134. $dz=\dfrac{y^2dx+x^2dy}{(x+y)^2}\operatorname{arctg}(xy+x+y)+\dfrac{xy[(y+1)dx+(x+1)dy]}{(x+y)[1+(xy+x+y)^2]}$.

3135. $\dfrac{e^{\frac{x^2+y^2}{xy}}}{x^2y^2}[(y^4-x^4+2xy^3)x\,dy+(x^4-y^4+2x^3y)y\,dx]$.

3136. $$\left.\begin{aligned}\frac{\partial z}{\partial x}&=2x\frac{\partial f}{\partial u}+ye^{xy}\frac{\partial f}{\partial v},\\ \frac{\partial z}{\partial y}&=-2y\frac{\partial f}{\partial u}+xe^{xy}\frac{\partial f}{\partial v}\end{aligned}\right\}\begin{aligned}u&=x^2-y^2;\\ v&=e^{xy}.\end{aligned}$$

3185. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{2x^2+y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{xy}{\sqrt{x^2+y^2}}$.

3186. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^3+(x^2-y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{\frac{3}{2}}(x+\sqrt{(x^2+y^2)^2}}$;

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}.$$

3187. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{2x}{(1+x^2)^2}$; $\dfrac{\partial^2 z}{\partial y^2}=-\dfrac{2y}{(1+y^2)^2}$; $\dfrac{\partial^2 z}{\partial x\partial y}=0$.

3188. $\dfrac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by)$; $\dfrac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$;

$$\frac{\partial^2 z}{\partial x\partial y}=2ab\cos 2(ax+by).$$

3189. $\dfrac{\partial^2 z}{\partial x^2}=e^{xe^y+2y}$; $\dfrac{\partial^2 z}{\partial y^2}=x(1+xe^y)e^{xe^y+y}$; $\dfrac{\partial^2 z}{\partial x\partial y}=(1+xe^y)e^{xe^y+y}$.

3190. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{4y}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{4x}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{2(x-y)}{(x+y)^3}$.

3191. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\ln y(\ln y+1)}{x^2}\,e^{\ln x\ln y}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\ln x(\ln x-1)}{y^2}\,e^{\ln x\ln y}$;

$\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\ln x\ln y+1}{xy}\,e^{\ln x\ln y}$.

3192. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{xy^3}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial^3 y}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{1}{\sqrt{(1-x^2y^2)^3}}$.

3193. $\dfrac{(x-z)\,y}{\sqrt{(x^2+y^2+z^2-2xz)^3}}$. **3194.** $2y^3(2+xy^2))\,e^{xy^2}$.

3195. $\dfrac{4x(3y^2-x^2)}{(x^2+y^2)^3}$. **3196.** $-x(2\sin xy+xy\cos xy)$.

3197. $(x^2y^2z^2+3xyz+1)\,e^{xyz}$.

3198. $mn(n-1)(n-2)p(p-1)\,x^{m-1}y^{n-3}z^{p-2}$. **3204.** $a=-3$.

3209. $$\frac{d^2y}{dx^2}=-\frac{\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{\partial f}{\partial y}\right)^2-2\dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}+\dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3}=\frac{1}{\left(\dfrac{\partial f}{\partial y}\right)^3}\begin{vmatrix}0 & \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y}\\ \dfrac{\partial f}{\partial y} & \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix}$$

3219. $-2y\,dx^2+4(y-x)\,dx\,dy+2x\,dy^2$. **3220.** $-\dfrac{(dx-dy)^2}{(x-y)^2}$.

3221. $\dfrac{(3x^2-y^2)\,dx^2+8xy\,dx\,dy+(3y^2-x^2)\,dy^2}{(x^2+y^2)^3}$.

3222. $2\sin 2y\,dx\,dy+2x\cos 2y\,dy^2$. **3223.** $e^{xy}[(y\,dx+y\,dy)^2+2\,d\,x\,dy]$.

3224. $2(z\,dx\,dy+y\,dx\,dz+x\,dy\,dz)$.

3225. $-\cos(2x+y)(2\,dx+dy)^3$; $(2\,dx+dy)^3$; 0.

3226. $-\sin(x+y+z)(dx+dy+dz)^2$.

3227. $-\dfrac{c^4}{z^3}\left[\left(\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}\right)\dfrac{dx^2}{a^2}+\dfrac{2xy}{a^2b^2}\,dx\,dy+\left(\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\dfrac{dy^2}{b^2}\right]$.

3228. $\dfrac{2z\,[xy^3\,dx^2+(x^2y^2+2xyz^2-z^4)\,dx\,dy+x^3y\,dy^2]}{(z^2-xy)^3}$.

3229. $-31{,}5\,dx^2+206\,dx\,dy-306\,dy^2$. **3230.** $\dfrac{d^2y}{dt^2}+y$.